Lecture: Corporate Income Tax - Unlevered firms

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Discounted Cash Flow, Section 3.1

Remark: The slightly expanded second edition (Springer, open access) has different enumeration than the first (Wiley). We use Springer's enumeration in the slides and Wiley's in the videos.

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Outline

The unlevered firm 1

Companies are indebted, i.e. levered. Why should we consider unlevered firms, i.e. firms without debt?

Valuation requires knowledge of

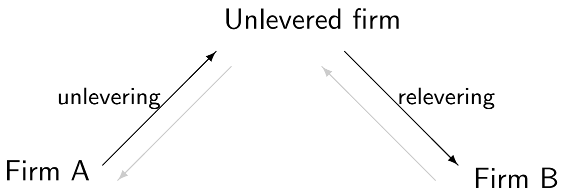
- cash flows $\Longleftarrow$ from business plans, annual balance sheets etc.
- taxes $\Longleftarrow$ from tax law
- cost of capital $\Longleftarrow$ from similar companies.

What is a 'similar company'?

Companies are similar with respect to

- business risk
- **financial** risk (= different leverage ratio).

We eliminate the financial risk by determining the cost of capital of an unlevered firm (unlevering) and then of a levered firm (relevering).



Notation 3

levered firm	index l
unlevered firm	index u
free cash flows (after taxes)	$\widetilde{FCF}_t^{\mathrm{u}}$
value	$\widetilde{V}_t^{\mathrm{u}}$
cost of capital	$k_t^{E,\mathrm{u}} =_{\mathsf{Def}} \frac{\mathsf{E}\left[\widetilde{FCF}_{t+1}^{\mathrm{u}} + \widetilde{V}_{t+1}^{\mathrm{u}} \mid \mathcal{F}_t\right]}{\widetilde{V}_t^{\mathrm{u}}} - 1$

We have, analogously to chapter 1 (even with the same proof!)

Theorem 3.1 (value of unlevered firm): *With deterministic cost of capital*

$$\widetilde{V}_t^{\mathrm{u}} = \sum_{s=t+1}^{T} \frac{\mathsf{E}\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{\left(1+k_t^{E,\mathrm{u}}\right)\cdots\left(1+k_{s-1}^{E,\mathrm{u}}\right)}.$$

An assumption is necessary, which was already used by Feltham/Christensen, Feltham/Ohlson, also everyday business in statistics: ‘autoregressive cash flows’.

Remark: This assumption will concern the stochastic structure of the unlevered cash flows.

Independence vs. uncorrelation

Autoregressive cash flows, also called AR(1):

$$\widetilde{FCF}^{\mathrm{u}}_{t+1} = (1+g)\widetilde{FCF}^{\mathrm{u}}_{t} + \widetilde{\varepsilon}_{t+1}.$$

In finance the usual assumption is: noise terms $\widetilde{\varepsilon}_t$ are pairwise **independent**. But here we only assume that the noise terms are pairwise **uncorrelated** – which is less restrictive:

$\widetilde{\varepsilon}_t, \widetilde{\varepsilon}_s$ are

uncorrelated if	independent if for all functions f and g
$\operatorname{Cov}\left[\widetilde{\varepsilon}_t, \widetilde{\varepsilon}_s\right] = 0$	$\operatorname{Cov}\left[f\left(\widetilde{\varepsilon}_t\right), g\left(\widetilde{\varepsilon}_s\right)\right] = 0$

Furthermore, our growth rate g_t can be time-dependent – that is why we speak of 'weak' autoregression.

Assumption 3.1 (weak autoregression): *There are growth rates (real numbers!) g_t such that for the cash flows of the unlevered firm*

$$\mathrm{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1}|\mathcal{F}_t\right] = (1+g_t)\widetilde{FCF}^{\mathrm{u}}_t.$$

Is this the same as definition AR(1) above? Yes, and this will be shown using our rules!

Define noise by

$$\widetilde{\varepsilon}_{t+1} := \widetilde{FCF}^{\mathrm{u}}_{t+1} - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}.$$

Then we can show

1. **Noise has no expectation**

$$\mathrm{E}\left[\widetilde{\varepsilon}_t\right] = 0.$$

2. **Noise terms are uncorrelated**

$$\mathrm{Cov}\left[\widetilde{\varepsilon}_s, \widetilde{\varepsilon}_t\right] = 0 \qquad \text{if } s \neq t$$

Noise terms have no expectation:

$$
\begin{aligned}
\mathsf{E}\left[\widetilde{\varepsilon}_{t+1}\right] &= \mathsf{E}\left[\widetilde{\varepsilon}_{t+1}|\mathcal{F}_0\right] && \text{by rule 1}\\
&= \mathsf{E}\left[\mathsf{E}\left[\widetilde{\varepsilon}_{t+1}|\mathcal{F}_t\right]|\mathcal{F}_0\right] && \text{by rule 4}\\
&= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1} - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_t\right]|\mathcal{F}_0\right] && \text{by definition}\\
&= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1}|\mathcal{F}_t\right] - \mathsf{E}\left[(1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_t\right]|\mathcal{F}_0\right] && \text{by rule 2}\\
&= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1}|\mathcal{F}_t\right] - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_0\right] && \text{by rule 5}\\
&= \mathsf{E}\left[(1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t} - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_0\right] && \text{by assumption}\\
&= 0 && \text{QED}
\end{aligned}
$$

Noise terms are uncorrelated: $(s < t)$

$$
\begin{aligned}
\operatorname{Cov}\left[\widetilde{\varepsilon}_s, \widetilde{\varepsilon}_t\right] &= \mathsf{E}\left[\widetilde{\varepsilon}_s \widetilde{\varepsilon}_t\right] - \underbrace{\mathsf{E}\left[\widetilde{\varepsilon}_s\right] \mathsf{E}\left[\widetilde{\varepsilon}_t\right]}_{=0} && \text{by definition of covariance} \\
&= \mathsf{E}\left[\widetilde{\varepsilon}_s \widetilde{\varepsilon}_t\right] \\
&= \mathsf{E}\left[\widetilde{\varepsilon}_s \widetilde{\varepsilon}_t | \mathcal{F}_0\right] && \text{by rule 1} \\
&= \mathsf{E}\left[\mathsf{E}\left[\widetilde{\varepsilon}_s \widetilde{\varepsilon}_t | \mathcal{F}_s\right] | \mathcal{F}_0\right] && \text{by rule 4} \\
&= \mathsf{E}\left[\widetilde{\varepsilon}_s \underbrace{\mathsf{E}\left[\widetilde{\varepsilon}_t | \mathcal{F}_s\right]}_{\substack{\text{will now shown} \\ \text{to be 0}}} | \mathcal{F}_0\right] && \text{by rule 5} \\
&= 0 && \text{QED}
\end{aligned}
$$

Proof (2) continued

$$
\begin{aligned}
\mathrm{E}\left[\tilde{\varepsilon}_t|\mathcal{F}_s\right] &= \mathrm{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1} - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_s\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1} - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_t\right]|\mathcal{F}_s\right] && \text{rule 4} \\
&= \mathrm{E}\left[\mathrm{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t+1}|\mathcal{F}_t\right] - (1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_s\right] && \text{rule 2 and 5} \\
&= 0 && \text{assumption 2.1.}
\end{aligned}
$$

Implications

What follows from weak autoregressive cash flows?
Two important theorems:

1. There is a deterministic dividend–price ratio.
2. The cost of capital of the unlevered firm may be used as a discount rate.

Theorem 3.2 (Williams, Gordon–Shapiro, Feltham/Ohlson): *If costs of capital are deterministic and cash flows are weak autoregressive, then*

$$\widetilde{V}_t^{\mathrm{u}} = \frac{\widetilde{FCF}_t^{\mathrm{u}}}{d_t^u}$$

holds for a deterministic dividend–price ratio d_t^u.

(Our first multiple!)

First notice that ($s > t$)

$$\begin{aligned}
\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_s|\mathcal{F}_t\right] &= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_s|\mathcal{F}_{s-1}\right]|\mathcal{F}_t\right] && \text{by rule 4}\\
&= \mathsf{E}\left[(1+g_{s-1})\widetilde{FCF}^{\mathrm{u}}_{s-1}|\mathcal{F}_t\right] && \text{by assumption 3.1}\\
&= (1+g_{s-1})\,\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_{s-1}|\mathcal{F}_t\right] && \text{by rule 2}\\
&= (1+g_{s-1})\cdots(1+g_t)\,\mathsf{E}\left[\widetilde{FCF}^{\mathrm{u}}_{t}|\mathcal{F}_t\right] && \text{continued}\\
&= (1+g_{s-1})\cdots(1+g_t)\widetilde{FCF}^{\mathrm{u}}_{t} && \text{by rule 5.}
\end{aligned}$$

Proof of theorem 3.2 (continued)

$$\widetilde{V}_t^{\mathrm{u}} = \sum_{s=t+1}^{T} \frac{\mathsf{E}\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{\left(1+k_t^{E,\mathrm{u}}\right)\cdots\left(1+k_{s-1}^{E,\mathrm{u}}\right)} \qquad \text{from Theorem 2.1}$$

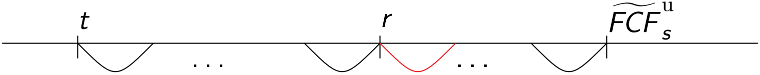
$$= \underbrace{\sum_{s=t+1}^{T} \frac{(1+g_{s-1})\cdots(1+g_t)}{\left(1+k_t^{E,\mathrm{u}}\right)\cdots\left(1+k_{s-1}^{E,\mathrm{u}}\right)}}_{:=1/d_t^u} \widetilde{FCF}_t^{\mathrm{u}} \qquad \text{see slide above}$$

$$= \frac{\widetilde{FCF}_t^{\mathrm{u}}}{d_t^u} \qquad \text{QED}$$

We want to look at discount rates now. First let us precisely define them.

Notice that discount rates will depend

- on the cash flow we want to value ($\widetilde{FCF}_s^{\mathrm{u}}$),
- on the point in time where we determine this value (index t) and
- on the actual time period (index r) we are discounting.



We use the notation $\kappa_r^{t\to s}$ for discounting from $r+1$ to r.

Definition 3.2 (discount rates): *Real numbers are called discount rates of the cash flow* $\widetilde{FCF}_t^{u}$ *if they satisfy*

$$\underbrace{\frac{\mathrm{E}_Q\left[\widetilde{FCF}_s^{\mathrm{u}}|\mathcal{F}_t\right]}{(1+r_f)^{s-t}}}_{\text{value}} = \frac{\mathrm{E}\left[\widetilde{FCF}_s^{\mathrm{u}}|\mathcal{F}_t\right]}{(1+\kappa_t^{t\to s})\cdots\left(1+\kappa_{s-1}^{t\to s}\right)}.$$

Interpretation of rhs: the way we use discount rates.

Interpretation of lhs: value, follows from fundamental theorem.

Now, finally, our second implication from weak autocorrelated cash flows.

Theorem 3.3 (equivalence of valuation concepts): *If costs of capital are deterministic and cash flows are weak autoregressive, then*

$$\frac{\mathrm{E}_Q\left[\widetilde{FCF}^{\mathrm{u}}_s|\mathcal{F}_t\right]}{(1+r_f)^{s-t}} = \frac{\mathrm{E}\left[\widetilde{FCF}^{\mathrm{u}}_s|\mathcal{F}_t\right]}{\left(1+k_t^{E,\mathrm{u}}\right)\cdots\left(1+k_{s-1}^{E,\mathrm{u}}\right)}$$

or: costs of capital are discount rates!

Notice that **sums** are equal

$$\underbrace{\sum_{s=t+1}^{T} \frac{\mathrm{E}_Q\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{(1+r_f)^{s-t}}}_{\text{fundamental theorem}} = \widetilde{V}_t^{\mathrm{u}} = \underbrace{\sum_{s=t+1}^{T} \frac{\mathrm{E}\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{(1+k_t^{E,\mathrm{u}}) \cdots (1+k_{s-1}^{E,\mathrm{u}})}}_{\text{theorem 1.1}}.$$

$$4 + 6 = 10 = 3 + 7$$

Theorem 3.3 tells us that **summands** are equal as well

$$\frac{\mathrm{E}_Q\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{(1+r_f)^{s-t}} = \frac{\mathrm{E}\left[\widetilde{FCF}_s^{\mathrm{u}} | \mathcal{F}_t\right]}{(1+k_t^{E,\mathrm{u}}) \cdots (1+k_{s-1}^{E,\mathrm{u}})}.$$

$4 \neq 3$ and $6 \neq 7$

This result is not trivial at all!

Proof of Theorem 3.3

The shining of the proof. . .

We skip the proof!

We assume that $k^{E,\mathrm{u}} = 20\%$. The expectations are

$$\mathsf{E}\left[\widetilde{FCF}_1^{\mathrm{u}}\right] = 100, \quad \mathsf{E}\left[\widetilde{FCF}_2^{\mathrm{u}}\right] = 110, \quad \mathsf{E}\left[\widetilde{FCF}_3^{\mathrm{u}}\right] = 121.$$

The value of the firm is given by

$$\begin{aligned} V_0^{\mathrm{u}} &= \frac{\mathsf{E}\left[\widetilde{FCF}_1^{\mathrm{u}}\right]}{(1+k^{E,\mathrm{u}})} + \frac{\mathsf{E}\left[\widetilde{FCF}_2^{\mathrm{u}}\right]}{(1+k^{E,\mathrm{u}})^2} + \frac{\mathsf{E}\left[\widetilde{FCF}_3^{\mathrm{u}}\right]}{(1+k^{E,\mathrm{u}})^3} \\ &= \frac{100}{1+0.2} + \frac{110}{(1+0.2)^2} + \frac{121}{(1+0.2)^3} \approx 229.75. \end{aligned}$$

Although not clear yet why necessary, we want to determine the market value at $t = 1$:

$$\mathrm{E}\left[\widetilde{FCF}_2^{\mathrm{u}} | \mathcal{F}_1\right] = \begin{cases} 121 & \text{up,} \\ 99 & \text{down.} \end{cases} \qquad \mathrm{E}\left[\widetilde{FCF}_3^{\mathrm{u}} | \mathcal{F}_1\right] = \begin{cases} 133.1 & \text{up,} \\ 108.9 & \text{down.} \end{cases}$$

hence

$$\left.\begin{aligned} \widetilde{V}_1^{\mathrm{u}}(u) &= \frac{\mathrm{E}\left[\widetilde{FCF}_2^{\mathrm{u}}(u)\right]}{1+k^{E,\mathrm{u}}} + \frac{\mathrm{E}\left[\widetilde{FCF}_3^{\mathrm{u}}(u)\right]}{(1+k^{E,\mathrm{u}})^2} \\ &= \frac{121}{1+0.2} + \frac{133.1}{(1+0.2)^2} \\ &\approx 193.26 \\ \widetilde{V}_1^{\mathrm{u}}(d) &\approx 158.13 \end{aligned}\right\} \Rightarrow \widetilde{V}_1^{\mathrm{u}} = \begin{cases} 193.26 & \text{up,} \\ 158.13 & \text{down.} \end{cases}$$

Let $r_f = 10\%$. Another additional result is the determination of the risk-neutral probability Q. Due to theorem 3.3 (or: costs of capital are discount rates) we have

$$\frac{\mathrm{E}_Q\left[\widetilde{FCF}_3^{\mathrm{u}}|\mathcal{F}_2\right]}{1+r_f} = \frac{\mathrm{E}\left[\widetilde{FCF}_3^{\mathrm{u}}|\mathcal{F}_2\right]}{1+k^{E,\mathrm{u}}} \; .$$

Assume that state ω occurred at time $t=2$. Then this equation translates to

Determining Q (continued)

$$\frac{\mathsf{E}_Q\left[\widetilde{FCF}_3^{\mathrm{u}}|\mathcal{F}_2\right]}{1+r_f}=\frac{Q_3(u|\omega)\,\widetilde{FCF}_3^{\mathrm{u}}(u|\omega)+Q_3(d|\omega)\,\widetilde{FCF}_3^{\mathrm{u}}(d|\omega)}{1+r_f}$$

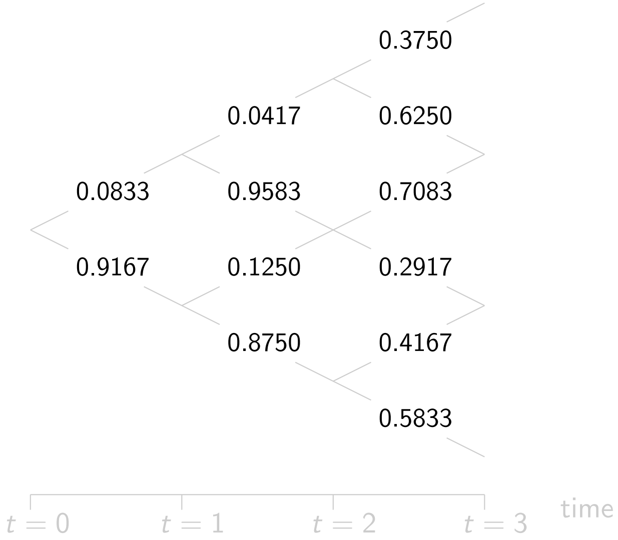
$$=\frac{P_3(u|\omega)\,\widetilde{FCF}_3^{\mathrm{u}}(u|\omega)+P_3(d|\omega)\,\widetilde{FCF}_3^{\mathrm{u}}(d|\omega)}{1+k^{E,\mathrm{u}}}=\frac{\mathsf{E}\left[\widetilde{FCF}_3^{\mathrm{u}}|\mathcal{F}_2\right]}{1+k^{E,\mathrm{u}}}\ .$$

Also, the conditional Q-probabilities add to one:

$$Q_3(u|\omega)+Q_3(d|\omega)=1\ .$$

This is a 2×2-system that can be solved for every ω!

Q in the finite example



As above Q can be determined:

$$Q_{t+1}(u|\omega) = \frac{\frac{1+r_f}{1+k^{E,\mathrm{u}}} - d}{u-d}, \qquad Q_{t+1}(d|\omega) = \frac{u - \frac{1+r_f}{1+k^{E,\mathrm{u}}}}{u-d}$$

regardless of t and ω.

Remark: The factors u and d cannot be chosen arbitrarily in the infinite case if the cost of capital $k^{E,\mathrm{u}}$ is given, because

$$d < \frac{1+r_f}{1+k^{E,\mathrm{u}}} < u$$

must hold in order to ensure positive Q-probabilities.

With $k^{E,\mathrm{u}} = 20\%$ the value of the unlevered firm is

$$V_0^{\mathrm{u}} = \frac{\mathrm{E}[\widetilde{FCF}_1^{\mathrm{u}}]}{k^{E,\mathrm{u}}} = 500.$$

We will consider unlevered and levered firms.

Cash flows of the unlevered firm are weak autoregressive, i.e. noise terms are uncorrelated.

The costs of capital of unlevered firm are discount rates.

The multiple dividend–price ratio is deterministic.